

On the Coefficients of a Hyperbolic Hydrodynamic Model

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Abstract

Based on the Nakajima-Zubarev type nonequilibrium density operator, we derive a hyperbolic hydrodynamical equation. Microscopic Kubo-formulas for all coefficients in the hyperbolic hydrodynamics are obtained. Coefficients α_i 's and β_i 's in the Israel-Stewart equation are given as current-weighted correlation lengths which are to be calculated in statistical mechanics.

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I. INTRODUCTION

The hydrodynamic model is one of the widely applied phenomenological models for relativistic heavy ion collisions. According to the recent detailed analyses, weak but non-vanishing viscosity is significant for the qualitative discussion of v_2 at RHIC [1]. Because the Navier-Stokes equation is a parabolic type equation, naive relativistic extension is not consistent with causality, and as a result, the numerical solution becomes unstable [2]. The relativistic causal hydrodynamic model which is a hyperbolic type had been introduced phenomenologically by Israel and Stewart more than thirty years ago [3, 4]. In the past decades, many proposals appear for the appropriate equations of the relativistic causal hydrodynamic model [5–7]. Most of the works seem to belong to the semi-classical phenomenological approach based on the Boltzmann equation. In this paper we derive a hyperbolic type hydrodynamics as a second order hydrodynamics based on a nonequilibrium density operator method [8]. Romatschke and Moore et al. also discussed hydrodynamics based on a nonequilibrium entropy density [9, 10]. In the present paper, more explicit formulas for the coefficients are issued.

The hydrodynamic model is a phenomenological model of the macroscopic point of view which is based on the coarse graining. The total system is constructed as a patchwork of a local system corresponding the statistical systems in equilibrium. The local system at each space-time point is expected to be microscopically large but macroscopically one point and moving with four velocity U^μ . Thermodynamical quantities and the coefficients in the hydrodynamical equation are the local quantities as the functions of position through the local thermodynamical parameters such as temperature $T(x)$ and chemical potential $\mu(x)$. The macroscopic model based on the coarse graining is justified by the fact that the microscopic correlation length is much smaller than the typical scale of the macroscopic dynamics. More than fifty years ago, Iso et al. discussed the applicability condition of the hydrodynamic model for the multiple production in high energy pp collisions based on the comparison between the length of the canonical correlation of the current operators and the scale of the change in the solution of the hydrodynamical equation [11].

Regarding the self-consistent macroscopic model, before we start to solve the hydrodynamical equation, all functional forms of the coefficients should be fixed. All coefficients of the hydrodynamical equation are given as functions of thermodynamical parameters, T and

μ . The derivation of them is a major task for statistical mechanics. The well-known Kubo formulas for the viscosities and heat conductivity provide us ways to evaluate the coefficients of the Navier-Stokes equation by statistical mechanics in equilibrium. However, the method to calculate the additional coefficients in the hyperbolic hydrodynamical equation, the so called Israel-Stewart equation, have not yet been established. As far as the authors know, only the kinetic calculation by using the Grad 14 moments method is adopted. The aim of this paper is to establish a way to calculate all coefficients in the hyperbolic hydrodynamics as the functions of temperature and chemical potential. Based on the nonequilibrium density operator method, we systematically derive microscopic formulas for the coefficients α_i 's and β_i 's in the Israel-Stewart equation. All coefficients are expressed as the current-weighted length of canonical correlation which are to be calculated in statistical mechanics in equilibrium with T and μ . In general, evaluation of the canonical correlation is not easy, but our formulas can be good targets for the future Lattice QCD simulations and the hadro-molecular calculation.

II. HYDRODYNAMIC MODEL

Hydrodynamical equation is composed of the energy-momentum conservation law

$$\partial^\mu T_{\mu\nu} = 0 \quad (1)$$

and the charge conservation law

$$\partial^\mu J_\mu = 0. \quad (2)$$

The flow of the fluid is described by a four velocity U^μ which is a normalized time-like vector, $U^\mu U_\mu = 1$. By using U^μ , the space-like projection operator is defined as $\Delta^{\mu\nu} = g^{\mu\nu} - U^\mu U^\nu$. With U^μ and $\Delta^{\mu\nu}$, we can define four projection operators of the second rank symmetric tensor:

$$\mathcal{T}^{(\mu\nu|\rho\sigma)} = \frac{1}{2} \left(\Delta^{\mu\rho} \Delta^{\nu\sigma} + \Delta^{\mu\sigma} \Delta^{\nu\rho} - \frac{2}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma} \right), \quad (3)$$

$$\mathcal{S}^{(\mu\nu|\rho\sigma)} = \frac{2}{3} \Delta^{\mu\nu} \Delta^{\rho\sigma}, \quad (4)$$

$$\mathcal{V}^{(\mu\nu|\rho\sigma)} = \frac{1}{2} (U^\mu U^\rho \Delta^{\nu\sigma} + \Delta^{\mu\rho} U^\nu U^\sigma + U^\mu U^\sigma \Delta^{\nu\rho} + \Delta^{\mu\sigma} U^\nu U^\rho), \quad (5)$$

$$\mathcal{U}^{(\mu\nu|\rho\sigma)} = U^\mu U^\nu U^\rho U^\sigma. \quad (6)$$

$\mathcal{T}(\mu\nu|\rho\sigma)$ stands for the traceless part of the spatial component, $\mathcal{S}(\mu\nu|\rho\sigma)$ stands for the trace of the spatial component, $\mathcal{U}(\mu\nu|\rho\sigma)$ stands for the time-like time-like component, and $\mathcal{V}(\mu\nu|\rho\sigma)$ stands for the time-like space-like or the space-like time-like component, respectively. These operators form a complete set

$$\mathcal{S}(\mu\nu|\rho\sigma) + \mathcal{T}(\mu\nu|\rho\sigma) + \mathcal{U}(\mu\nu|\rho\sigma) + \mathcal{V}(\mu\nu|\rho\sigma) = \frac{1}{2}(g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) = \mathbb{1}. \quad (7)$$

U^μ corresponds to the direction of the time on the local rest frame at x^μ and $\Delta^{\mu\nu}$ specifies the space-like coordinate, respectively. Hence, on the local rest frame, U^μ becomes $(1, 0, 0, 0)$ and $\Delta^{\mu\nu}$ becomes $-\delta^{ij}$, respectively. Throughout this paper, Greek indexes, μ, ν, ρ, σ etc. stand for the Lorentzian $(0, 1, 2, 3)$, and Latin indexes i, j, k etc. stand for the Euclidian $(1, 2, 3)$, respectively. A space-time point (\mathbf{x}, t) is denoted as x^μ for the abbreviation. $\tau = x^\mu U_\mu$ stands for the time on the local rest frame (comoving frame of the fluid element), and $D = U^\mu \partial_\mu$ stands for the derivative by τ , respectively.

Energy density is given as the time-like time-like component of the energy-momentum tensor and pressure is given as the the trace of the spatial part, respectively,

$$\varepsilon = T^{\mu\nu} U_\mu U_\nu, \quad (8)$$

$$P = \frac{1}{3} T^{\mu\nu} \Delta_{\mu\nu}. \quad (9)$$

Once Lagrangian of the system is given, the energy-momentum tensor $T^{\mu\nu}$ is defined as the space-time translation operator. However, the definition of the flow U^μ of a relativistic fluid is not trivial. Eckart [12] and Namiki et al. [13] define U^μ based on the particle flow (P-frame) and Weinberg used charge current [14]. In the textbook by Landau and Lifshitz, U^μ is defined as a eigenvector of $T^{\mu\nu}$ [15]. Recently another possible choice is proposed by Tsumura et al. [16]. In this paper we adopt the Landau-Lifshitz frame (E-frame) where U^μ is an eigenvector of the energy momentum tensor:

$$T^{\mu\nu} U_\nu = \varepsilon U^\mu. \quad (10)$$

Stress-shear tensor is given as the spatial traceless part of the energy momentum tensor:

$$\pi^{\mu\nu} = \mathcal{T}(\mu\nu|_{\rho\sigma}) T^{\rho\sigma}. \quad (11)$$

Using the projection operators, we can denote energy density as:

$$\varepsilon = U_\mu U_\nu \mathcal{U}(\mu\nu|_{\rho\sigma}) T^{\rho\sigma}, \quad (12)$$

and pressure as:

$$P = \frac{1}{3} \Delta_{\mu\nu} \mathcal{S}^{(\mu\nu|\rho\sigma)} T^{\rho\sigma}, \quad (13)$$

respectively. The time-like space-like components of the energy momentum tensor should correspond to the heat flow, however, because of Eq. (10) these components must vanish:

$$\mathcal{V}^{(\mu\nu|\rho\sigma)} T^{\rho\sigma} = 0, \quad (14)$$

on the E-frame.

The charge density n and the charge current J^μ are given by $n = J^\mu U_\mu$ as usual, however, U^μ is proportional to the energy flow on the E-frame, and the charge current contains I^μ which is perpendicular to U^μ :

$$J^\mu = nU^\mu + I^\mu, \quad (15)$$

$$I^\mu = \Delta^\mu_\nu J^\nu. \quad (16)$$

I^μ corresponds to the *heat flow on the E-frame* [15].

III. NONEQUILIBRIUM DENSITY OPERATOR

The nonequilibrium density operator is given as [17–19]:

$$\hat{\rho} = Q^{-1} \exp(-\hat{A} + \hat{B}), \quad (17)$$

with \hat{A} being:

$$\hat{A} = \int dS^\mu \left\{ \beta(\mathbf{x}, t) U^\nu(\mathbf{x}, t) \hat{T}_{\mu\nu}(\mathbf{x}, t) - \beta(\mathbf{x}, t) \mu(\mathbf{x}, t) \hat{J}_\mu(\mathbf{x}, t) \right\}, \quad (18)$$

\hat{B} being:

$$\begin{aligned} \hat{B} = \lim_{\zeta \rightarrow 0^+} \int d^4 x' e^{-\zeta U^\mu(x-x')_\mu} \left\{ \hat{T}_{\mu\nu}(\mathbf{x}', t') \partial'^\mu (\beta(\mathbf{x}', t') U^\nu(\mathbf{x}', t')) \right. \\ \left. + \hat{J}_\nu(\mathbf{x}', t') \partial'^\nu (\beta(\mathbf{x}', t') \mu(\mathbf{x}', t')) \right\} \Bigg], \quad U^\mu(x-x')_\mu \geq 0 \end{aligned} \quad (19)$$

and with Q being normalization $Q = \text{tr}[\exp(-\hat{A} + \hat{B})]$. The expectation value of an operator $\hat{O}(x)$ is given by $\langle \hat{O}(x) \rangle = \text{tr}[\hat{\rho} \hat{O}(x)]$.

\hat{A} corresponds to the local equilibrium density operator, $\hat{\rho}_l = Q_l^{-1} \exp(-\hat{A})$, with $Q_l^{-1} = \text{tr}[\exp(-\hat{A})]$ being normalization. dS^μ is a three-dimensional hypersurface which locally

corresponds to the comoving volume element of the fluid. The so-called matching conditions, $\langle \hat{\varepsilon} \rangle = \langle \hat{\varepsilon} \rangle_l$ and $\langle \hat{n} \rangle = \langle \hat{n} \rangle_l$, are imposed for $\beta(\mathbf{x}, t)$ and $\beta(\mathbf{x}, t)\mu(\mathbf{x}, t)$ in $\hat{\rho}_l$. If the expectation value of some operator $\hat{O}(x^\mu)$:

$$\langle \hat{O}(x^\mu) \rangle_l = Q_l^{-1} \text{tr}[\hat{\rho}_l \hat{O}(x^\mu)], \quad (20)$$

is a local quantity in the macroscopic sense, on the local rest frame at that point, the expectation value is the same to the one in the corresponding equilibrium state where the inverse temperature is $1/T = \beta(\mathbf{x}, t)$ and the chemical potential is $\mu(\mathbf{x}, t)$, respectively.

\hat{B} stands for the influence of the thermodynamical forces such as the disturbances in the flow and the thermodynamical parameters. The adiabatic limit $\zeta \rightarrow 0+$ should be taken at the final stage of the calculation.

According to the standard procedure of the linear response theory, we expand the nonequilibrium density operator $\hat{\rho}$ at around the local equilibrium operator $\hat{\rho}_l$ up to the linear term of the thermodynamical forces. Expectation value $\langle \hat{O}(x) \rangle$ is given as a canonical correlation with the force term:

$$\begin{aligned} \langle \hat{O}(x) \rangle &= \langle \hat{O}(x) \rangle_l + \lim_{\zeta \rightarrow 0+} \int d^4x' e^{-\zeta U^\mu(x-x')_\mu} \left(\hat{O}(x), \hat{T}_{\rho\sigma}(x') \partial'^\rho (\beta(x') U^\sigma(x')) \right) \\ &\quad + \lim_{\zeta \rightarrow 0+} \int d^4x' e^{-\zeta U^\mu(x-x')_\mu} \left(\hat{O}(x), \hat{J}_\rho(x') \partial'^\rho (\beta(x') \mu(x')) \right), \end{aligned} \quad (21)$$

with (\hat{O}_1, \hat{O}_2) being:

$$(\hat{O}_1, \hat{O}_2) = \langle \hat{O}_1 \int_0^1 d\lambda e^{\lambda A} \hat{O}_2 e^{-\lambda A} \rangle_l - \langle \hat{O}_1 \rangle_l \langle \hat{O}_2 \rangle_l. \quad (22)$$

IV. DERIVATIVE EXPANSION WITH THE THERMODYNAMICAL FORCES

If the change of the thermodynamical parameters and the flow are slow enough and can be treated as almost constant during the microscopic relaxation length which is determined by the canonical correlation Eq. (22), we may adopt the Taylor expansion of the thermodynamical forces $\partial'^\rho (\beta(x') U^\sigma(x'))$ and $\partial'^\rho (\beta(x') \mu(x'))$ at around the position x of the operator $\hat{O}(x)$, and put the forces outside of the integration in Eq. (21). Then we can obtain expectation values of $T^{\mu\nu}$ and J^μ in the power series of the gradient of the thermodynamical

parameters and the flow:

$$\begin{aligned}
\langle \hat{T}^{\mu\nu}(x) \rangle &= \langle \hat{T}^{\mu\nu}(x) \rangle_l + \left(\int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{-\zeta(t-t')} (\hat{T}^{\mu\nu}(x), \hat{T}_{\rho\sigma}(x')) \right) \partial^\rho (\beta U^\sigma) \\
&+ \left(\int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{-\zeta(t-t')} (\hat{T}^{\mu\nu}(x), \hat{J}_\rho(x')) \right) \partial^\rho (\beta \mu) \\
&+ \left(\int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{-\zeta(t-t')} (\hat{T}^{\mu\nu}(x), (x_\lambda' - x_\lambda) \hat{T}_{\rho\sigma}(x')) \right) \partial^\lambda \partial^\rho (\beta U^\sigma) \\
&+ \left(\int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{-\zeta(t-t')} (\hat{T}^{\mu\nu}(x), (x_\lambda' - x_\lambda) \hat{J}_\rho(x')) \right) \partial^\lambda \partial^\rho (\beta \mu), \quad (23)
\end{aligned}$$

$$\begin{aligned}
\langle \hat{J}^\mu(x) \rangle &= \langle \hat{J}^\mu(x) \rangle_l + \left(\int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{-\zeta(t-t')} (\hat{J}^\mu(x), \hat{J}_\rho(x')) \right) \partial^\rho (\beta \mu) \\
&+ \left(\int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{-\zeta(t-t')} (\hat{J}^\mu(x), \hat{T}_{\rho\sigma}(x')) \right) \partial^\rho (\beta U^\sigma) \\
&+ \left(\int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{-\zeta(t-t')} (\hat{J}^\mu(x), (x_\lambda' - x_\lambda) \hat{J}_\rho(x')) \right) \partial^\lambda \partial^\rho (\beta \mu) \\
&+ \left(\int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{-\zeta(t-t')} (\hat{J}^\mu(x), (x_\lambda' - x_\lambda) \hat{T}_{\rho\sigma}(x')) \right) \partial^\lambda \partial^\rho (\beta U^\sigma). \quad (24)
\end{aligned}$$

Here all coefficients are given as the integral of the canonical correlation with the local equilibrium density operator (22). If the non-vanishing region of the canonical correlation in the integrand are limited only in the microscopic scale, we can regard them as local quantities in the macroscopic sense and we may substitute a statistical mechanical calculation in equilibrium with the corresponding T and μ for the expectation with ρ_l in Eqs. (23) and (24).

The first terms in the right hand side of (23) and (24), which are free from the thermodynamical forces, correspond to the perfect fluid part: $\langle \hat{T}^{\mu\nu}(x) \rangle_l = (\langle \hat{\varepsilon} \rangle_l + \langle \hat{P} \rangle_l) U^\mu U^\nu - \langle \hat{P} \rangle_l g^{\mu\nu}$ and $\langle \hat{J}^\mu(x) \rangle_l = \langle \hat{n} \rangle_l U^\mu$. The second terms in the right hand side of (23) and (24), which are the the zeroth order gradient terms of thermodynamical forces, correspond to the Navier-Stokes equation. The Kubo-formulas for viscosities and heat conductivity are obtained as the coefficients in these terms [13, 17, 18]. The third terms in the right hand side of (23) and (24), which are the the first order gradient terms of thermodynamical forces, correspond to the hyperbolic hydrodynamical equation [8].

Decomposing $\hat{T}^{\mu\nu}$ into the space-like components and the time-like components, we can

rewrite \hat{B} as:

$$\begin{aligned}\hat{B} = \lim_{\zeta \rightarrow 0+} \int d^4x' e^{-\zeta U^\mu(x-x')_\mu} \{ & \beta(\mathbf{x}', t') \hat{\pi}_{\mu\nu}(\mathbf{x}', t') \partial'^\mu (U^\nu(\mathbf{x}', t')) \\ & - \hat{I}_\nu(\mathbf{x}', t') \partial'^\nu (\beta(\mathbf{x}', t') \mu(\mathbf{x}', t')) \\ & - \beta(\mathbf{x}', t') \hat{P}'(\mathbf{x}', t') \partial_\mu U^\mu(\mathbf{x}', t') \},\end{aligned}\quad (25)$$

where:

$$\hat{P}' = \hat{P} - \left(\frac{\partial \langle P \rangle_l}{\partial \langle \varepsilon \rangle_l} \right)_{\langle n \rangle_l} \hat{\varepsilon} - \left(\frac{\partial \langle P \rangle_l}{\partial \langle n \rangle_l} \right)_{\langle \varepsilon \rangle_l} \hat{n}.\quad (26)$$

By virtue of the matching condition and the E-frame condition (10), $\mathcal{S}^{(\mu\nu} |_{\rho\sigma}) T^{\rho\sigma}$ and $\mathcal{V}^{(\mu\nu} |_{\rho\sigma}) T^{\rho\sigma}$ don't appear in \hat{B} .

Let us denote integration of the canonical correlation in Eqs. (23), (24) as:

$$\langle \langle O_1 | O_2 \rangle \rangle = \lim_{\zeta \rightarrow 0+} \int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{-\zeta(t-t')} \left(\hat{O}_1(\mathbf{x}, t), \hat{O}_2(\mathbf{x}', t') \right),\quad (27)$$

for the abbreviation. Assuming isotropy of the local rest system, we can apply Curie's theorem for $\langle \langle O_1 | O_2 \rangle \rangle$. $\langle \langle O_1 | O_2 \rangle \rangle$ depends on (\mathbf{x}, t) only through the temperature and the chemical potential. Because of the translational invariance of the corresponding equilibrium system, we can rewrite:

$$\lim_{\zeta \rightarrow 0+} \int d^3\mathbf{x}' \int_{-\infty}^t dt' e^{-\zeta(t-t')} \left(\hat{O}_1(\mathbf{x}, t), (x'^\mu - x^\mu) \hat{O}_2(\mathbf{x}', t') \right),\quad (28)$$

$$= \lim_{\zeta \rightarrow 0+} \int d^3\mathbf{x} \int_{-\infty}^0 dt e^{+\zeta t} \left(\hat{O}_1(\mathbf{0}, 0), x^\mu \hat{O}_2(\mathbf{x}, t) \right),\quad (29)$$

$$= \langle \langle O_1 | x^\mu | O_2 \rangle \rangle.\quad (30)$$

Substituting the projection operator $g^{\mu\nu} = U^\mu U^\nu + \Delta^{\mu\nu}$ into equations (23) and (24), and applying the Curie's theorem, we can obtain simple forms of the expectation values of the

thermodynamical currents:

$$\begin{aligned}
\langle \hat{\pi}^{\mu\nu} \rangle &= \langle \hat{\pi}^{\mu\nu} \rangle_l + \beta \langle \langle \pi^{\mu\nu} | \pi^{\rho\sigma} \rangle \rangle \partial_\rho U_\sigma - \langle \langle \pi^{\mu\nu} | I^\rho \rangle \rangle \partial_\rho (\beta\mu) \\
&\quad + \beta \langle \langle \pi^{\mu\nu} | x^\lambda | \pi^{\rho\sigma} \rangle \rangle \partial_\lambda \partial_\rho U_\sigma - \langle \langle \pi^{\mu\nu} | x^\lambda | I^\rho \rangle \rangle \partial_\lambda \partial_\rho (\beta\mu) \\
&= \langle \hat{\pi}^{\mu\nu} \rangle_l + \beta \langle \langle \pi^{\mu\nu} | \pi^{\rho\sigma} \rangle \rangle \partial_\rho U_\sigma \\
&\quad + \beta \langle \langle \pi^{\mu\nu} | \tau | \pi^{\rho\sigma} \rangle \rangle D \partial_\rho U_\sigma - \langle \langle \pi^{\mu\nu} | x_\lambda | I^\rho \rangle \rangle \Delta^{\lambda\kappa} \partial_\kappa \partial_\rho (\beta\mu), \tag{31}
\end{aligned}$$

$$\begin{aligned}
\langle \hat{I}^\mu \rangle &= \langle \hat{I}^\mu \rangle_l - \beta \langle \langle I^\mu | I^\nu \rangle \rangle \partial_\nu (\beta\mu) - \langle \langle I^\mu | x^\rho | I^\nu \rangle \rangle \partial_\rho \partial_\nu (\beta\mu) \\
&\quad + \beta \langle \langle I^\mu | x^\nu | \pi^{\rho\sigma} \rangle \rangle \partial_\nu \partial_\rho U_\sigma - \beta \langle \langle I^\mu | x^\nu | P' \rangle \rangle \partial_\nu \partial_\rho U_\rho \\
&= \langle \hat{I}^\mu \rangle_l - \beta \langle \langle I^\mu | I^\nu \rangle \rangle \Delta_\nu{}^\rho \partial_\rho (\beta\mu) - \langle \langle I^\mu | \tau | I^\nu \rangle \rangle D \partial_\nu (\beta\mu) \\
&\quad + \beta \langle \langle I^\mu | x^\nu | \pi^{\rho\sigma} \rangle \rangle \Delta_\nu{}^\kappa \partial_\kappa \partial_\rho U_\sigma - \beta \langle \langle I^\mu | x^\nu | P' \rangle \rangle \Delta_\nu{}^\sigma \partial_\sigma \partial_\rho U_\rho, \tag{32}
\end{aligned}$$

$$\begin{aligned}
\langle \hat{P} \rangle &= \langle \hat{P} \rangle_l - \beta \langle \langle P | P' \rangle \rangle \partial_\nu U^\nu \\
&\quad - \beta \langle \langle P | x^\rho | P' \rangle \rangle \partial_\rho \partial_\nu U^\nu - \langle \langle P | x^\rho | I^\sigma \rangle \rangle \partial_\rho \partial_\nu (\beta\mu) \\
&= \langle P \rangle_l - \beta \langle \langle P | P' \rangle \rangle \partial_\nu U^\nu \\
&\quad - \beta \langle \langle P | \tau | P' \rangle \rangle D \partial_\nu U^\nu - \langle \langle P | x^\rho | I^\sigma \rangle \rangle \Delta_\rho{}^\sigma \partial_\sigma \partial_\nu (\beta\mu). \tag{33}
\end{aligned}$$

For the isotropic system, we can factorize scalar coefficients from tensor structures as follows:

$$\langle \langle I^i | I^j \rangle \rangle = \langle I | I \rangle \delta^{ij}, \tag{34}$$

$$\langle \langle \pi^{ij} | \pi^{kl} \rangle \rangle = \frac{\langle \pi | \pi \rangle}{2} \left(\delta^{ik} \delta^{jk} + \delta^{il} \delta^{jk} - \frac{2}{3} \delta^{ij} \delta^{kl} \right), \tag{35}$$

$$\langle \langle \pi^{ij} | \tau | \pi^{kl} \rangle \rangle = \frac{\langle \pi | \mathbf{t} | \pi \rangle}{2} \left(\delta^{ik} \delta^{jk} + \delta^{il} \delta^{jk} - \frac{2}{3} \delta^{ij} \delta^{kl} \right), \tag{36}$$

$$\langle \langle I^i | \tau | I^j \rangle \rangle = \langle I | \mathbf{t} | I \rangle \delta^{ij}, \tag{37}$$

$$\langle \langle P | P' \rangle \rangle = \langle P | P' \rangle, \tag{38}$$

$$\langle \langle P | \tau | P' \rangle \rangle = \langle P | \mathbf{t} | P' \rangle, \tag{39}$$

$$\langle \langle I^i | x^j | P' \rangle \rangle = \langle I | \mathbf{x} | P' \rangle \delta^{ij}, \tag{40}$$

$$\langle \langle \pi^{ij} | x^k | I^l \rangle \rangle = \frac{\langle \pi | \mathbf{x} | I \rangle}{2} \left(\delta^{ik} \delta^{jk} + \delta^{il} \delta^{jk} - \frac{2}{3} \delta^{ij} \delta^{kl} \right), \tag{41}$$

where scalar coefficients in the right hand side are defined by:

$$\langle \pi | \pi \rangle = \frac{1}{5} \langle \langle \pi^{\mu\nu} | \pi_{\mu\nu} \rangle \rangle, \quad (42)$$

$$\langle \pi | \mathbf{t} | \pi \rangle = \frac{1}{5} \langle \langle \pi^{\mu\nu} | \tau | \pi_{\mu\nu} \rangle \rangle, \quad (43)$$

$$\langle I | I \rangle = \frac{-1}{3} \langle \langle I^\mu | I_\mu \rangle \rangle, \quad (44)$$

$$\langle I | \mathbf{t} | I \rangle = \frac{-1}{3} \langle \langle I^\mu | \tau | I_\mu \rangle \rangle, \quad (45)$$

$$\langle \pi | \mathbf{x} | I \rangle = \frac{1}{5} \langle \langle \pi^{\mu\nu} | x_\mu | I_\nu \rangle \rangle, \quad (46)$$

$$\langle I | \mathbf{x} | P' \rangle = \frac{-1}{3} \Delta^{\mu\nu} \langle \langle I_\mu | x_\nu | P' \rangle \rangle, \quad (47)$$

$$\langle P | \mathbf{x} | I \rangle = \frac{-1}{3} \Delta^{\mu\nu} \langle \langle P | x_\nu | I_\mu \rangle \rangle. \quad (48)$$

Both in equations (31) and (32), because of the isotropy, the first term in the right hand side should vanish:

$$\langle \pi^{\mu\nu} \rangle_l = 0, \quad (49)$$

$$\langle I^\mu \rangle_l = 0. \quad (50)$$

$\langle \hat{P} \rangle_l$ is static pressure. The second terms in the right hand side of Eqs. (31) and (33) stand for the Kubo-formulas for the shear viscosity η_s and the bulk viscosity η_v :

$$\eta_s = \beta \langle \pi | \pi \rangle, \quad (51)$$

$$\eta_v = \beta \langle P | P' \rangle. \quad (52)$$

The coefficient in the second term in the right hand side of Eq. (32) corresponds to the heat conductivity κ on the E-frame[15]:

$$\kappa = \left(\frac{\langle \hat{\varepsilon} \rangle_l + \langle \hat{P} \rangle_l}{\langle \hat{n} \rangle_l T} \right)^2 \langle I | I \rangle. \quad (53)$$

Finally, we can rewrite equations (31),(32) and (33) as:

$$\begin{aligned}\langle \hat{\pi}^{\mu\nu} \rangle &= \eta_s \mathcal{T}^{(\mu\nu|\rho\sigma)} \partial_\rho U_\sigma + \beta \langle \pi | \mathbf{t} | \pi \rangle \mathcal{T}^{(\mu\nu|\rho\sigma)} D \partial_\rho U_\sigma \\ &\quad - \langle \pi | \mathbf{x} | I \rangle \mathcal{T}^{(\mu\nu|\rho\sigma)} \partial_\rho \partial_\sigma (\beta\mu),\end{aligned}\tag{54}$$

$$\begin{aligned}\langle \hat{I}^\mu \rangle &= \kappa \left(\frac{\langle \hat{n} \rangle_l T}{\langle \hat{\varepsilon} \rangle_l + \langle \hat{P} \rangle_l} \right)^2 \Delta^{\mu\nu} \partial_\nu (\beta\mu) + \langle I | \mathbf{t} | I \rangle \Delta^{\mu\nu} D \partial_\nu (\beta\mu) \\ &\quad + \beta \langle I | \mathbf{x} | \pi \rangle \mathcal{T}^{(\mu\nu|\rho\sigma)} \partial_\nu \partial_\rho U_\sigma + \beta \langle I | \mathbf{x} | P' \rangle \Delta^{\mu\nu} \partial_\nu \partial_\sigma U^\sigma,\end{aligned}\tag{55}$$

$$\langle \hat{P} \rangle - \langle \hat{P} \rangle_l = -\eta_v \partial_\mu U^\mu - \beta \langle P | \mathbf{t} | P' \rangle D \partial_\mu U^\mu + \langle P | \mathbf{x} | I \rangle \Delta^{\mu\nu} \partial_\mu \partial_\nu (\beta\mu).\tag{56}$$

Substituting the first order equation (the Navier-Stokes equation):

$$\begin{aligned}\langle \hat{\pi}^{\mu\nu} \rangle &= \eta_s \mathcal{T}^{(\mu\nu|\rho\sigma)} \partial_\rho U_\sigma, \\ \langle \hat{I}^\mu \rangle &= \kappa \left(\frac{\langle \hat{n} \rangle_l T}{\langle \hat{\varepsilon} \rangle_l + \langle \hat{P} \rangle_l} \right)^2 \Delta^{\mu\nu} \partial_\nu (\beta\mu), \\ \langle \hat{P} \rangle - \langle \hat{P} \rangle_l &= -\eta_v \partial_\mu U^\mu,\end{aligned}$$

for the second order derivative terms of U^μ and $\beta\mu$, we can obtain the Israel-Stewart equation:

$$\begin{aligned}\langle \hat{\pi}^{\mu\nu} \rangle &= \eta_s \mathcal{T}^{(\mu\nu|\rho\sigma)} \partial_\rho U_\sigma + \frac{\beta \langle \pi | \mathbf{t} | \pi \rangle}{\eta_s} \mathcal{T}^{(\mu\nu|\rho\sigma)} D \langle \hat{\pi}^{\rho\sigma} \rangle \\ &\quad - \frac{\langle \pi | \mathbf{x} | I \rangle}{\kappa \left(\frac{\langle \hat{n} \rangle_l T}{\langle \hat{\varepsilon} \rangle_l + \langle \hat{P} \rangle_l} \right)^2} \mathcal{T}^{(\mu\nu|\rho\sigma)} \partial_\rho \langle \hat{I}_\sigma \rangle,\end{aligned}\tag{57}$$

$$\begin{aligned}\langle \hat{I}^\mu \rangle &= \kappa \left(\frac{\langle \hat{n} \rangle_l T}{\langle \hat{\varepsilon} \rangle_l + \langle \hat{P} \rangle_l} \right)^2 \Delta^{\mu\nu} \partial_\nu (\beta\mu) + \frac{\langle I | \mathbf{t} | I \rangle}{\kappa \left(\frac{\langle \hat{n} \rangle_l T}{\langle \hat{\varepsilon} \rangle_l + \langle \hat{P} \rangle_l} \right)^2} \Delta^{\mu\nu} D \langle \hat{I}_\nu \rangle \\ &\quad + \frac{\beta \langle I | \mathbf{x} | \pi \rangle}{\eta_s} \mathcal{T}^{(\mu\nu|\rho\sigma)} \partial_\nu \langle \hat{\pi}_{\rho\sigma} \rangle + \frac{\beta \langle I | \mathbf{x} | P' \rangle}{-\eta_v} \Delta^{\mu\nu} \partial_\nu \left(\langle \hat{P} \rangle - \langle \hat{P} \rangle_l \right),\end{aligned}\tag{58}$$

$$\langle \hat{P} \rangle - \langle \hat{P} \rangle_l = -\eta_v \partial_\mu U^\mu + \frac{\beta \langle P | \mathbf{t} | P' \rangle}{\eta_v} D \left(\langle \hat{P} \rangle - \langle \hat{P} \rangle_l \right) + \frac{\langle P | \mathbf{x} | I \rangle}{\kappa \left(\frac{\langle \hat{n} \rangle_l T}{\langle \hat{\varepsilon} \rangle_l + \langle \hat{P} \rangle_l} \right)^2} \Delta^{\mu\nu} \partial_\mu \langle \hat{I}_\nu \rangle.\tag{59}$$

Comparing the above equations (57), (58) and (59) with (2.38) in ref [4], we can identify the α_i 's and β_i 's in the Israel-Stewart equation in our notation. Besides thermodynamical

quantities and the numerical factors coming from the differences in the definition, β_i 's in ref. [4] are current-weighted correlation times; $\beta_0 \sim \langle P|\mathbf{t}|P'\rangle$, $\beta_1 \sim \langle I|\mathbf{t}|I\rangle$ and $\beta_2 \sim \langle \pi|\mathbf{t}|\pi\rangle$. α_i in ref. [4] is cross-current-weighted correlation length; $\alpha_1 \sim \langle I|\mathbf{x}|\pi\rangle$. As concerns bulk current, though the original current operator is \hat{P} the thermodynamical current in \hat{B} is \hat{P}' , hence, $\alpha_0 \sim \langle P|\mathbf{x}|I\rangle$ in bulk current (2.38a) and $-\alpha_0 \sim \langle I|\mathbf{x}|P'\rangle$ in charge current (2.38b) in ref. [4].

Normalizing the "state" $|\pi\rangle$, $|I\rangle$ and $|P'\rangle$ by using the relation, (51), (52) and (53), we define the current-weighted correlation times:

$$\bar{\tau}_I = \frac{\langle I|\mathbf{t}|I\rangle}{\langle I|I\rangle}, \quad (60)$$

$$\bar{\tau}_s = \frac{\langle \pi|\mathbf{t}|\pi\rangle}{\langle \pi|\pi\rangle}, \quad (61)$$

$$\bar{\tau}_v = \frac{\langle P|\mathbf{t}|P'\rangle}{\langle P|P'\rangle}, \quad (62)$$

and the cross-current weighted correlation distances:

$$\bar{\mathbf{x}}_{Is} = \left(\frac{\langle \hat{\varepsilon} \rangle_l + \langle \hat{P} \rangle_l}{\langle \hat{n} \rangle_l T} \right) \sqrt{\frac{1}{\kappa}} \sqrt{\frac{\beta}{\eta_s}} \langle I|\mathbf{x}|\pi \rangle, \quad (63)$$

$$\bar{\mathbf{x}}_{Iv} = \left(\frac{\langle \hat{\varepsilon} \rangle_l + \langle \hat{P} \rangle_l}{\langle \hat{n} \rangle_l T} \right) \sqrt{\frac{1}{\kappa}} \sqrt{\frac{\beta}{\eta_v}} \langle I|\mathbf{x}|P' \rangle, \quad (64)$$

$$\bar{\mathbf{x}}_{vI} = \left(\frac{\langle \hat{\varepsilon} \rangle_l + \langle \hat{P} \rangle_l}{\langle \hat{n} \rangle_l T} \right) \sqrt{\frac{1}{\kappa}} \sqrt{\frac{\beta}{\eta_v}} \langle P|\mathbf{x}|I \rangle. \quad (65)$$

All these quantities are the functions of the temperature T and the chemical potential μ , which are able to be calculated based on the statistical mechanics in equilibrium.

V. CONCLUDING REMARKS

Based on the nonequilibrium density operator, we have derived the hyperbolic hydrodynamical equation. All coefficients in the equations are expressed as the integration of the canonical correlations of the current operators which are to be calculated in statistical mechanics. Our discussion is nothing special for the relativity but we simply expand the thermodynamical forces in their derivatives. The additional coefficients in the Israel-Stewart equation α_i 's and β_i 's are given as the current-weighted correlation times and the cross-current-weighted correlation distances.

The current-weighted correlation time β_i corresponds to the relaxation time of the current. The evaluation of the relaxation time β_i is essentially the same to the calculation of the Kubo-formulas for the transport coefficient[20]. According to the hadro-molecular simulation, each current exhibits its own relaxation even if the basic dynamics is common and once we succeed to figure out the behavior of the relaxation of the currents, we can easily evaluate both the relaxation time β_i and the transport coefficient[21]. This method has also been applied to the calculation of the viscosity of a hadron gas[22, 23]. α_i 's are the correlation distances between different currents which have not yet been investigated.

As Iso et.al. discussed, the comparison between the microscopic correlation length and macroscopic scale in hydrodynamics is the touch-stone of the model [11]. The canonical correlations exhibit typical microscopic scales of the system. The evaluation of α_i and β_i or quantities in Eqs. (62)~(67) based on the Lattice QCD, or hadro-molecular calculation, will provide us the key informations to justify the hydrodynamic model of the QCD matter. We have treated only the linear term of the thermodynamical forces, but higher order perturbation or non-perturbative treatment would be appreciated in the microscopic calculation of the canonical correlation.

The bulk current, Eq. (26) is a natural extension of the \hat{P}' in [19] and [18] where the charge conservation is neglected. If a conserved current exists, $\hat{P} \propto \hat{\varepsilon}$ is not enough to make \hat{P}' vanish. Therefore, even if $\langle \hat{P} \rangle = \frac{1}{3} \langle \hat{\varepsilon} \rangle$ is achieved in a high energy region, the vanishing of the bulk viscosity may not be trivial.

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